Lecture 08: Applications of Talagrand Inequality

Convex Distance

- Last lecture we considered $d_H(x, y) := \left|\{i\colon x_i \neq y_i\}\right|$
- We generalize this notion of distance to $\alpha \in \mathbb{R}^n$ such that each of its coordinates are $\geqslant 0$ and $\|\alpha\| = 1$, i.e., $\sum_{i=1}^n \alpha_i^2 = 1$

Definition

$$d_\alpha(x, y) := \sum_{\substack{1 \leqslant i \leqslant n \\ x_i \neq y_i}} \alpha_i$$

- We define the convex distance as

Definition (Convex Distance)

$$d_T(x, y) := \sup_{\alpha\colon \|\alpha\|=1} \sum_{\substack{1 \leqslant i \leqslant n \\ x_i \neq y_i}} \alpha_i$$

Talagrand Inequality

We will not prove this inequality. We simply state it and shall use it to show the concentrated behavior of the longest increasing subsequence.

Theorem (Talagrand Inequality)

$$\mathbb{P}\left[\mathbb{X} \in A\right] \cdot \mathbb{P}\left[d_T(\mathbb{X}, A) \geqslant t\right] \leqslant \exp\left(-t^2/4\right)$$

Longest Increasing Subsequence

- Suppose $x = (x_1, \ldots, x_n)$ and each coordinate has been sampled independently and uniformly at random over $\mathbb{R}$
- Let $f(x)$ represent the longest increasing subsequence of x
- Suppose $f(x) = k$
- Note that there exists $K_x \subseteq \{1, \ldots, n\}$ such that the longest increasing subsequence is formed by

$$(x_{i_i}, x_{i_2}, \ldots, x_{i_k})$$

- Note that if y matches x everywhere in K_x, then we must have $f(y) \geqslant f(x)$
- Similarly, if y matches x everywhere in K_x except at ℓ positions, then $f(y) \geqslant f(x) - \ell$
- In particular, we can write the following bound:

$$f(y) \geqslant f(x) - \left|\{i \colon i \in K_x, x_i \neq y_i\}\right|$$

- Let α be a vector such that $\alpha_i = 0$ if $i \notin K_x$, otherwise $\alpha_i = 1/\sqrt{f(x)}$. Note that $\|\alpha\| = 1$ and the above inequality can equivalently be written as:

$$f(y) \geqslant f(x) - \sqrt{f(x)} d_\alpha(x, y)$$

- Rearranging, we get $f(x) \leqslant f(y) + \sqrt{f(x)}d_\alpha(x,y)$
- By the definition of $d_T(\cdot,\cdot)$, we can also write that:

$$f(x) \leqslant f(y) + \sqrt{f(x)}d_T(x,y)$$

- Let $A_a = \{y \colon f(y) \leqslant a\}$
- Consider $y \in A_a$. The above inequality becomes:

$$f(x) \leqslant a + \sqrt{f(x)}d_T(x,y)$$

- Rearranging, we get:

$$d_T(x,y) \geqslant \frac{f(x) - a}{\sqrt{f(x)}}$$

- Suppose $f(x) \geqslant a + t$. Then, it implies that

$$d_T(x, y) \geqslant \frac{t}{\sqrt{a+t}}$$

- Therefore, we can conclude that

$$\mathbb{P}\left[f(\mathbb{X}) \geqslant a + t\right] \leqslant \mathbb{P}\left[d_T(\mathbb{X}, y) \geqslant \frac{t}{\sqrt{a+t}}\right]$$

- Since, this statement is true for all $y \in A_a$, the following statement is also true

$$\mathbb{P}\left[f(\mathbb{X}) \geqslant a + t\right] \leqslant \mathbb{P}\left[d_T(\mathbb{X}, A_a) \geqslant \frac{t}{\sqrt{a+t}}\right]$$

- Multiplying both sides by $\mathbb{P}[\mathbb{X} \in A_a]$, we get

$$\begin{aligned}
&\mathbb{P}[\mathbb{X} \in A_a] \cdot \mathbb{P}\left[f(\mathbb{X}) \geqslant a + t\right] \\
\leqslant &\mathbb{P}[\mathbb{X} \in A_a] \cdot \mathbb{P}\left[d_T(\mathbb{X}, A_a) \geqslant \frac{t}{\sqrt{a+t}}\right] \\
\leqslant &\exp\left(-t^2/4(a+t)\right)
\end{aligned}$$

- Suppose we choose $a = m$, where m is the median of the distribution $f(\mathbb{X})$. So, we have $\mathbb{P}\left[f(\mathbb{X}) \leqslant m\right] \geqslant 1/2$. We get:

$$\mathbb{P}\left[f(\mathbb{X}) \geqslant m + t\right] \leqslant 2\exp\left(-t^2/4(m+t)\right)$$

- Suppose we choose $a = m - t$. Then $\mathbb{P}\left[f(\mathbb{X}) \geqslant a + t\right] \geqslant 1/2$. Now we have:

$$\mathbb{P}\left[\mathbb{X} \in A_a\right] = \mathbb{P}\left[f(\mathbb{X}) \leqslant m - t\right] \leqslant 2\exp\left(-t^2/4m\right)$$

Configuration Function

Definition (Configuration Function)

A function f is a c-configuration function, if for every x, y, there exists α such that the following holds.

$$f(y) \geqslant f(x) - \sqrt{c \cdot f(x)}\, d_\alpha(x, y)$$

Note that the longest increasing subsequence function is an 1-configuration function. The derivation of the previous concentration bound on the longest increasing subsequence naturally generalizes to c-configuration functions.